One Mass-Radius relation for Planets, White Dwarfs and Neutron Stars

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Abstract

We produce the simple formula for the radius, R, of a cold body:

$$\left(\tfrac{4}{3}\pi\rho_o\right)^{1/3} R = M_p^{1/3}\frac{y}{1+y^2}I$$

where $y^3 = M/M_p$, M is the total mass, $J = \sqrt{1 - y^4/y_{ch}^4}$ and $I = J/\left[1 + \tfrac{3}{4}\left(1 - J^{\frac{1}{2}}\right)\right]$.

$y_{ch}^3 = M_{ch}/M_p$ where M_{ch} is Chandrasekhar's limiting mass. The density of our liquid/solid matter at low pressure defines the constant ρ_o. For bodies with non relativistic electrons $I = J = 1$.

M_p is constant and gives the mass of the planet of maximum radius. M_{ch}, M_p and ρ_o are all defined in terms of fundamental constants, in particular the ratio M_p/M_{ch} is basically given by the three halves power of the fine structure constant.

The aim of the discussion is to emphasise physical principles and to connect the masses derived to the fundamental constants, so mathematical niceties are sacrificed and replaced by interpolations between simple exact limits.

1 Introduction

The virial theorem shows that a star deprived of any energy sources will shrink and its kinetic energy will grow which normally implies it will get hotter. In the 1920's Eddington (1926) was puzzled as to how a star could ever cool down and end its life. This problem, together with the mystery of white dwarfs such as Sirius B, was beautifully solved by R. H. Fowler (1926) who realised that the electrons would become degenerate, as in a metal, so the star could then be supported by the degeneracy pressure which is an inevitable consequence of the zero-point energy combined with the Pauli exclusion principle.

Even bodies of zero temperature have such a pressure which supports the white dwarfs against gravity. Stoner (1929) made early models of white dwarfs to compare their densities with the theory but Anderson (1929) rightly criticised him for not allowing for the relativistic motions of electrons forced to very high densities. He showed that when this was allowed for, there would be a limiting mass beyond which no cold white dwarf models could exist. His estimate of the limiting mass was not very good, but Stoner (1930) then modified his own calculations and calculated the limit for homogeneous models which gave an answer good to 10%. Meanwhile, Chandrasekhar (1931) had developed the theory independently starting from Fowler's work. He realised that the limiting configuration would be an $n = 3$ polytrope and gave the first accurate determination of the Chandrasekhar limit.

More detailed models of white dwarfs were given by Salpeter (1967) who derived conditions for their solidification and determined properties of the lattice. Shapiro and Teukolsky in their book give a definitive account of how all this extends to neutron stars.

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Here our aim is much more modest. To give a rough approximate formula for the radius of any cold body as a function of its mass. We illustrate how the balance between degeneracy pressure, the electrical attraction of electrons to atomic nuclei, and gravitation lead to a mathematical formula that describes the mass-radius relationship for cold bodies. This holds from asteroids, through planets and brown dwarfs to white dwarfs and neutron stars. Understanding rather than accuracy is our aim so we shall use interpolation formulae where they are simpler, nevertheless we shall ensure accuracy in important limiting cases. Thus we aspire to follow the fine tradition set in Salpeter's 1967 lectures.

At masses below that of Saturn, the degeneracy pressure of the electrons within the atoms (vide the Thomas Fermi atom) is balanced by the electrical binding. While the gravity serves to hold the whole body in a nearly spherical shpae, it is not strong enough to crush the atoms into significantly smaller volumes. Thus the material is nearly incompressible and apart from any chemical fractionation the lighter bodies have the same density. However, above the mass of Saturn the gravitational potential energy increases to become comparable with the electrical one. Above the mass of Jupiter, the gravitation dominates, the atoms are crushed together with the outer electrons eventually bcoming pressure ionised to make a continuous sea. In fact, somewhat above Jupiter's mass, the weight of each layer added so compresses the material within that the whole body shrinks in radius. The electron sea is not so dense that the zero point energy drives the electrons to speeds close to that of light, so they give Fowler's $p = K\rho^{5/3}$ as an equation of state as in $n = 3/2$ polytropes. The $\int pdV$ energy behaves as $M^{5/3}/R^2$. Although the $-GM^2/R$ gravitational energy rises with a higher power of the mass, the total energy achieves its minimum at the (equilibrium) radius with $R \propto M^{-4/3}$. This is the brown dwarf white dwarf sequence. However, as M is further increased the increasing density of the bodies ($\bar{\rho} \propto M^2$) leads to zero point agitation with relativistic speeds. This results in the pressure density relationship $p = K'\rho^{4/3}$ with $\int pdV \propto M^{4/3}/R$. As this is approached the equilibrium radius rapidly becomes very small and very sensitive to mass because both gravitational and internal energies scale with the same power of the radius. No force balance is possible beyond Chandrasekhar's limiting mass.

In neutron stars the electrical energy is unimportant; most of the electrons have combined with protons and been transformed into neutrons so that neutron degeneracy supplies the pressure. Neutron star radii are smaller than white dwarf radii by the factor $\frac{m_e}{m_n}$ so the densities are $(m_n/m_e)^3$ times greater.

A secondary aim of this article is to express the important masses that arise in astronomy in terms of the fundamental physical constants. Thus the ratio of the Chandrasekhar mass to the mass of the planet of maximum radiu is essentially the fine structure constant raised to the power $-3/2$. Such matters are of particular importance to the misguided few who seriously entertain theories in which such fundamental constants change.

2   Kinetic Energy of Degenerate Material

The kinetic energy of an electron is given by

$$\epsilon = m_e c^2 \left[\sqrt{\frac{p^2}{m_e^2 c^2} + 1} - 1 \right] \quad . \tag{1}$$

The number of energy levels in a box of side L with momentum between p and $p + dp$ is

$$2 \times 4\pi p^2 dp L^3 h^{-3} \quad ,$$

where the initial 2 comes from the 2 spin states of an electron. If all levels up to momentum p_m are filled, the kinetic energy of the electrons per unit volume is

$$\mathcal{E} = \int_o^{p_m} 8\pi h^{-3} \epsilon p^2 dp \quad . \tag{2}$$

While the integral may be evaluated using $sh^{-1}\left(\frac{p_m}{m_e c}\right)$ that begins to obscure the physics so we shall evaluate the limiting cases and interpolate.

If $p_m \ll m_e c$ then $\epsilon = \frac{1}{2}p^2/m_e$ so

$$\mathcal{E} = \frac{4\pi}{5}h^{-3}p_m^5/m_e \quad .$$

(3)

If $p_m \gg m_e c$ then $\epsilon \simeq pc$ for most electrons, so

$$\mathcal{E} = 2\pi h^{-3}p_m^4 c \quad .$$

(4)

Looking at (1), an interpolation in terms of the function $\sqrt{x^2+1}-1$ where $x = \eta\frac{p_m}{m_e c}$ suggests itself, where η is to be determined. For large x this function gives $\eta p_m/(m_e c)$ while for small x it gives $\frac{1}{2}\eta^2\frac{p_m^2}{m_e^2 c^2}$.

So the ratio of the non-relativistic case to the very-relativistic case is $\frac{1}{2}\eta\frac{p_m}{m_e c}$. Now the ratio of the expressions (3) and (4) is $\frac{2}{5}\frac{p_m}{m_e c}$ so we choose $\eta = \frac{4}{5}$. The resulting interpolation formula exact at both limits is

$$\mathcal{E} = \frac{5}{2}\pi h^{-3}p_m^3 m_e c^2\left(\sqrt{x^2+1}-1\right) \quad ,$$

(5)

where

$$x = \frac{4}{5}\frac{p_m}{m_e c} \quad .$$

(6)

If the total mass per electron is $\mu_e m_H$ then the density is

$$\rho = \mu_e m_H \frac{8}{3}\pi p_m^3 h^{-3} = \rho_1 x^3 \quad ,$$

(7)

where

$$\rho_1 = \left(\frac{5}{2}\right)^3\frac{\pi\mu_e m_H}{3}/\left[h/(m_e c)\right]^3 \quad ,$$

(8)

thus ρ_1 is the density at which the electrons start to become relativistic due to the uncertainty principle. m_H is the mass per electron and $h/(m_e c)$ is its Compton wavelength. Rewriting (5) in terms of ρ

$$\mathcal{E}/\rho = \frac{15}{16}\frac{m_e c^2}{m_H \mu_e}\left(\sqrt{x^2+1}-1\right) \quad .$$

(9)

Now imagine expanding unit mass of this material

$$PdV = Pd\left(\frac{1}{\rho}\right) = -d\left(\mathcal{E}/\rho\right) = -d\left(\mathcal{E}/\rho\right)/dx \; dx \quad .$$

(10)

Since $\rho \propto x^3$ we find the pressure density relationship

$$P = P_1\frac{x^5}{\sqrt{x^2+1}} \quad ,$$

(11)

where

$$P_1 = \frac{5^4}{2^7}\frac{\pi}{3}\frac{m_e^4 c^5}{h^3} \quad .$$

(12)

In particular the non-relativistic and extreme relativistic limits give $P = K_\Gamma\rho^\Gamma$,

$$P = K_\Gamma\rho^\Gamma = \begin{cases} K_{5/3}\rho^{5/3} & \Gamma = 5/3 \quad \text{non} - \text{rel} \\ \\ K_{4/3}\rho^{4/3} & \Gamma = 4/3 \quad \text{extreme rel.} \end{cases}$$

(13)

From (11) and (7),

$$K_\Gamma = \begin{cases} P_1\rho_1^{-5/3} & \Gamma = 5/3 \\ \\ P_1\rho_1^{-4/3} & \Gamma = 4/3 \end{cases}$$

(14)

notice $P_1/\rho_1 = \frac{5}{16}\frac{m_e}{\mu_e m_H}c^2$.

Returning to (10) we see that whenever $\mathcal{E} \propto \rho^\Gamma$ then

$$P = (\Gamma - 1)\mathcal{E} \quad . \tag{15}$$

We have of course devised our derivation so that these formulae (8), (13), (14) & (15) are exact cf Shapiro & Teukolsky (1983).

Three limiting density distributions are important

1. The uniform density which we shall see occurs.

2. The non-relativistic $\Gamma = \frac{5}{3}$ polytrope of index $\frac{3}{2} = n = \frac{1}{\Gamma - 1}$.

3. The extreme relativistic polytrope of index $3, \Gamma = \frac{4}{3}$.

To orient ideas it is useful to consider initially the uniform density case. The kinetic energy is given by (9) & (7)

$$\mathrm{T} = \tfrac{4}{3}\pi \mathrm{R}^3 \mathcal{E} = \mathrm{A_o M} \left[\sqrt{(\zeta_o \mathrm{M}^{1/3}/\mathrm{R})^2 + 1} - 1 \right] \quad , \tag{16}$$

where we have written $x = \zeta_o M^{1/3}/R$ and $A_o = \frac{15}{16} \ \frac{m_e c^2}{m_H \mu_e} = \frac{3P_1}{\rho_1}$,

$$\zeta_o = \left(\tfrac{4}{3}\pi\rho_1\right)^{-1/3} \quad . \tag{17}$$

Although this uniform case is only realistic for very small x, nevertheless the modifications needed for polytropic cases occur only via structure constants of order unity. Thus much understanding can be gained by studying the problems with T given by (16) and (17) for all x. To show the modifications needed for the polytropic cases we now take the case of a polytrope of index $n = \frac{1}{\Gamma-1}$. Using (15)

$$\mathrm{T} = \int_o^\mathrm{R} 4\pi \mathrm{r}^2 \mathcal{E}\mathrm{dr} = n\mathrm{K}_\Gamma \mathrm{R}^3 \int_o^1 4\pi \left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2 \rho^\Gamma \mathrm{d}\left(\mathrm{r/R}\right) = n\mathrm{K}_\Gamma \mathrm{M} \left(\frac{3\mathrm{M}}{4\pi \mathrm{R}^3}\right)^{\Gamma-1} \mathrm{C_n} \quad ,$$

where

$$C_n = \int_o^1 3\left(\frac{r}{R}\right)^2 \left(\frac{\rho}{\bar\rho}\right)^{1+1/n} d\left(r/R\right) \quad , \tag{18}$$

is a structure constant for a polytrope of index n which is evaluated in the appendix: $C_o = 1, \quad C_{3/2} = 1.7501, \quad C_3 = 2.2146$. Notice that our uniform density case (16) also gives the $M(M/R^3)^{\Gamma-1}$ factor for both the non-relativistic $\Gamma = \frac{5}{3}$ and the extreme relativistic $\Gamma = \frac{4}{3}$ cases. Following our earlier interpolation procedure using η below equation (4), we now look for an interpolation suggested by (16) between the $\Gamma = \frac{5}{3}$ and $\Gamma = \frac{4}{3}$ polytropes. This is of the form

$$\mathrm{T} = \mathrm{AM} \left[\sqrt{\left(\frac{\zeta \mathrm{M}^{1/3}}{\mathrm{R}}\right)^2 + 1} - 1 \right] \quad , \tag{19}$$

where A and ζ are chosen to make the two polytropic formulae exact when $\zeta M^{1/3}/R \ll 1$ and $\gg 1$ respectively. This gives

$$\zeta = \frac{C_{3/2}}{C_3}\zeta_o = 0.7903\zeta_o; \quad A\zeta^2 = C_{3/2}A_o\zeta_o^2 = 1.7501 A_o \zeta_o^2 \quad . \tag{20}$$

Notice that (19) has precisely the same form as (16) but the values of A and ζ differ.

3 Potential Energy

The gravitational potential energy is $-\alpha GM^2/R$ where Ritter's formula tells us that for polytropes $\alpha = \frac{3}{5-n}$ (see appendix). However, for planets and small bodies the electrons are held in electrically rather than

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gravitationally. We must therefore include in the potential energy some estimate of the electrical potential energy. The mean number density of electrons is

$$n_e = M/\left(\tfrac{4}{3}\pi R^3 \mu_e m_H\right) \quad,$$

so we estimate the electrical potential energy as

$$-\beta\left(\frac{M}{\mu_e m_H}\right)\frac{e^2}{R\left(\frac{M}{\mu_e m_H}\right)^{-1/3}} \quad, \tag{21}$$

where the first bracket gives the number of electrons in the whole body and β is a dimensionless constant to be determined. The gravitational potential energy is proportional to M^2 while the electrical is proportional to $M^{4/3}$ at fixed R. Particular interest centres on the mass at which these two contributions are equal. Calling it M_1 we find with $\alpha = \tfrac{3}{5}$

$$M_1 = (5\beta/3)^{3/2}\,\mu_e^{-2}\left(\frac{e^2}{Gm_H^2}\right)^{3/2} m_H = \left(\frac{5\beta}{3}\right)^{3/2}\mu_e^{-2}2.308\times 10^{30} gm \quad. \tag{22}$$

Except for a structure factor of order unity M_1 is given by $\left(\frac{e^2}{Gm_p^2}\right)^{3/2}$, the ratio of the electrical to gravitational forces between two protons, raised to the $3/2$ power, times m_H. Notice that this mass does not depend on Planck's constant. In adapting formula (21) to rocky planets & asteroids we have used $\rho_o = 3.65$ and $\mu_e = 2$.

4 Energy Minimisation Gives Radius

The total energy is now given by

$$E = AM\left(\sqrt{\zeta^2 M^{2/3}R^{-2}+1}-1\right) - \alpha\frac{GM^2}{R}\left[1+\left(\frac{M_1}{M}\right)^{2/3}\right] \quad. \tag{23}$$

The equilibrium radius has the minimum E for any R so

$$\frac{R}{M}\ \frac{dE}{dR} = 0 \quad, \tag{24}$$

therefore
$$\frac{A\zeta^2 M^{2/3}R^{-2}}{\sqrt{\zeta^2 M^{2/3}R^{-2}+1}} = \frac{\alpha}{\zeta}\ \frac{\zeta M^{1/3}}{R}\ G\left(M^{2/3}+M_1^{2/3}\right) \quad, \tag{25}$$

equation (25) is readily solved for $\zeta M^{1/3}R^{-1}$ and thence for R

$$R = \frac{\zeta M^{1/3}}{q}\sqrt{1-q^2} \quad, \tag{26}$$

where

$$q = \frac{\alpha}{\zeta}\ \frac{G}{A}\left(M^{2/3}+M_1^{2/3}\right) \quad. \tag{27}$$

Evidently $R = 0$ when $q = 1$ so this gives Chandrasekhar's limiting mass. Since $M_1^{2/3}$ is negligible at this limit we have (putting $\alpha = \tfrac{3}{2}$ for the $n = 3$ polytrope $\Gamma = \tfrac{4}{3}$)

$$M_{ch} = \left(\frac{\zeta A}{\alpha G}\right)^{3/2} = \tfrac{3}{2}\sqrt{\pi}\left(\frac{C_3}{2}\right)^{3/2}\left(\frac{\hbar c}{Gm_H^2}\right)^{3/2}\frac{m_H}{\mu_e^2} \quad. \tag{28}$$

Which demonstrates that the Chandrasekhar limit is essentially the gravitational fine structure constant to the power of $-\tfrac{3}{2}$ times the mass of the hydrogen atom. Equation (26) when combined with (27) gives the mass-radius relationship

$$R = \frac{\zeta^2 A}{\alpha G}\ \frac{M^{1/3}}{M^{2/3}+M_1^{2/3}}\sqrt{1-\left(\frac{M}{M_{ch}}\right)^{4/3}} \quad, \tag{29}$$

where we have simplified the surd taking account of the fact the $M_{ch} \gg M_1$.

Equation (29) shows that there is a maximum radius for a cold planet close to M_1 i.e., close to Jupiter's mass. Also, for small M, R grows like $M^{1/3}$ so the density is constant. This is not surprising; the electrons are held in by electricity and we are merely placing together electrically bound neutral objects. As more mass is piled on the growth in R slows until close to M_1 the weight of the overlying material so crushes what is beneath that additional material hardly changes R. Beyond M_1 the crushing is so great that additional mass only serves to make R decrease like $M^{-1/3}$. This is the normal white dwarf regime which continues until the electrons are so confined that they become relativistic, the radius then shrinks precipitously as the Chandrasekhar limit is approached.

The same behaviour is predicted whether we use the crude homogeneous approximation of Stoner (1930) via (16) or the inhomogeneous ones via (19) but the detailed numbers will be different. The fact that for $M \ll M_1$ the bodies grow at constant density shows that they are not centrally condensed. Thus in that regime we should use the homogeneous model. However, for the non-relativistic white dwarfs we should use the polytrope of index $\frac{3}{2}$, $\Gamma = \frac{5}{3}$ and for the relativistic white dwarfs the polytrope of index 3, $\Gamma = \frac{4}{3}$. Each of these three cases is described by a different value of α which is $\frac{3}{5-n}$ with $n = 0$ corresponding to the homogeneous case. As n varies systematically with mass taking the values 0, $\frac{3}{2}$ and 3 in the low mass, white dwarf, and relativistic cases it is not difficult to devise and interpolation formula for α. However there is one further problem of this type. We chose ζ and A to fit the white dwarf density profiles but we now find that homogeneous profiles are appropriate for $M \ll M_1$. Thus we should interpolate not just α but rather $(\zeta^2 A/\alpha)$ which should take the value $\left[\zeta_o^2 A_o/(\frac{3}{5})\right]$ for small M and $\zeta^2 A/\alpha$ with $\alpha = \frac{3}{5-n}$ for the white dwarf regime. In the whole range the average density increases with mass so $\frac{\zeta^2 A}{\alpha G}/(M^{2/3} + M_1^{2/3})$ decreases as M increases (see 29).

The interpolation

$$\frac{\zeta^2 A}{\alpha G} = \frac{\zeta_o^2 A_o}{3/5G} \; \frac{M_1^{2/3} + M^{2/3}}{\left(M_1^{2/3} + 0.816 M^{2/3}\right) K} \quad , \tag{30}$$

where $K = 1 + \frac{3}{4}(1 - J^{1/2})$ and $J = (1 - y^4 y_{ch}^{-4})^{1/2}$, gives the correct values for M small and M large and ensures that the density increases. We now define $M_p = \left[M_1^{2/3}/0.816\right]^{3/2} = 1.356 M_1$. Then writing $y = (M/M_p)^{1/3}$, our interpolation formula (31) becomes

$$\frac{\zeta^2 A}{\alpha G} = \frac{5\zeta_o^2 A_o}{3G} \; \frac{M_1^{2/3} + M^{2/3}}{M_1^{2/3}(1 + y^2) K} \quad , \tag{31}$$

and our Mass radius relationship is

$$\left(\tfrac{4}{3}\pi\rho_o\right)^{1/3} R = M_p^{1/3} \frac{y}{1 + y^2} I \quad , \tag{32}$$

with $y = \left(\frac{M}{M_p}\right)^{1/3}$ and $y_{ch} = \left(\frac{M_{ch}}{M_p}\right)^{1/3}$ and $I = J/K$,

$$\left(\tfrac{4}{3}\pi\rho_o\right)^{1/3} = \beta \tfrac{5}{3} \left(\tfrac{4}{9\pi}\right)^{2/3} \mu_e^{1/3} \left(\frac{m_H}{a_o^3}\right)^{1/3} \quad ; \quad a_o = \frac{\hbar^2}{m_e e^2}$$

so ρ_o is apart from structural constants the density of hydrogen within the Bohr radius, a_o, of the nucleus. ρ_o depends on $\hbar$ through a_o.

$$M_p = 1.356 \, M_1 = 1.356 \left(\tfrac{5}{3}\beta\right)^{3/2} \mu_e^{-2} \left(\frac{e^2}{Gm_H^2}\right)^{3/2} m_H \quad . \tag{33}$$

This is the mass of the cold planet of maximum radius. As stated it is essentially the fine structure constant to the $\frac{3}{2}$ power times the Chandrasekhar mass or alternatively the ratio of electrical to gravitational forces between two protons, raised to the three halves power, times the hydrogen atom's mass.

5   The Mass Radius Relationship

Our mass-radius relationship is now defined except for the constant β that we inserted in (21)) to allow for the extreme crudeness of our estimate of the electrical potential energy. This we shall evaluate by fitting our formula to the radius of Saturn. Away from the relativistic regime formula (32) takes the even simpler form

$$\left(\tfrac{4}{3}\pi\rho_o\right)^{1/3} R = M_p^{1/3}\,\frac{y}{1+y^2} = \frac{M^{1/3}}{1+y^2} \quad . \tag{34}$$

Now both $\rho_o^{1/3}$ and y^{-2} are proportional to β so for a planet of known M and R we solve for β using Saturn

$$\beta = \left(\frac{\rho}{\rho_o\beta^{-3}}\right)^{1/3} - \left(\beta y^2\right) = 1.343 - \cdot230 = 1.113 \quad , \tag{35}$$

where in spite of appearances each bracketed term is independent of β and $\rho = M/\left(\tfrac{4}{3}\pi R^3\right)$. The same calculation for Jupiter gives $\beta = 1.161$ so $\beta = 1.137$ is a good compromise differing from each by only 2.1 per cent. This value of β yields $\rho_o = 0.419 gm\ cm^{-3}$ and $M_p = 6.24 \times 10^{30} gm$. The density of Hydrogen at the relatively low pressure (compared with planetary interiors) of half a Megabar is close to $0.54\ gm\ cm^{-3}$ (Alavi et al., 1995). At much lower pressures solid Hydrogen and liquid Helium have densities of 0.07 and $0.12\ gm\ cm^{-3}$. For Terrestial planets β (and hence $\rho_o^{1/3}$ and $M_p^{2/3}$) takes higher values appropriate to their composition. From (34) we see that the planet of maximum radius has a density of $8\rho_o$. Thus although M_p is independent of $\hbar$ nevertheless $R_{\max}$ depends on $\hbar$.

To summarise our basic result is that the radius R of a cold body of mass M is given by

$$\left(\tfrac{4}{3}\pi\rho_o\right)^{1/3} R = M_p^{1/3}\frac{y}{1+y^2}\ \ \frac{\sqrt{1 - y^4 y_{ch}^{-4}}}{\left\{1 + \tfrac{3}{4}\left[1 - \left(\sqrt{1 - y^4 y_{ch}^{-4}}\right)^{1/2}\right]\right\}} \quad , \tag{36}$$

where the surd and the curly bracketed expression reduce to 1 for non-relativistic white dwarfs and small bodies.

According to (36) the cold Planet of maximum radius has a mass of 3.3 times Jupiter's and its radius is 8.65×10^9cms as opposed to Jupiter's radius of 7.0×10^9 (after correction at constant volume for the eccentricity). Equation (36) gives 7.0×10^9 for the radius of a body of mass $M_J = 1.899 \times 10^{30}$gm

6   Neutron Stars

In the above we have not aimed to treat the mass radius relationship for neutron stars but with suitable modification the same principles apply to them.

I It is the degenerate neutrons that provide the support so m_e must be replaced by m_n wherever it occurs. The electric term is small and irrelevant. The mass per neutron is now m_n so μ_e is replaced by 1. In particular under (14) $p_1/\rho_1 = \tfrac{5}{16}c^2$ and in (8) ρ_1 is bigger by the factor $\left(\frac{m_n}{m_e}\right)^3$.

II The Chandrasekhar mass is almost unchanged except that we are now interested in $\mu_e = 1$ rather than the normal 2.

III General Relativistic corrections are no longer so small, so the effective gravity is somewhat stronger.

IV The forces between neutrons are such that the free neutron gas is no longer a very good approximation.
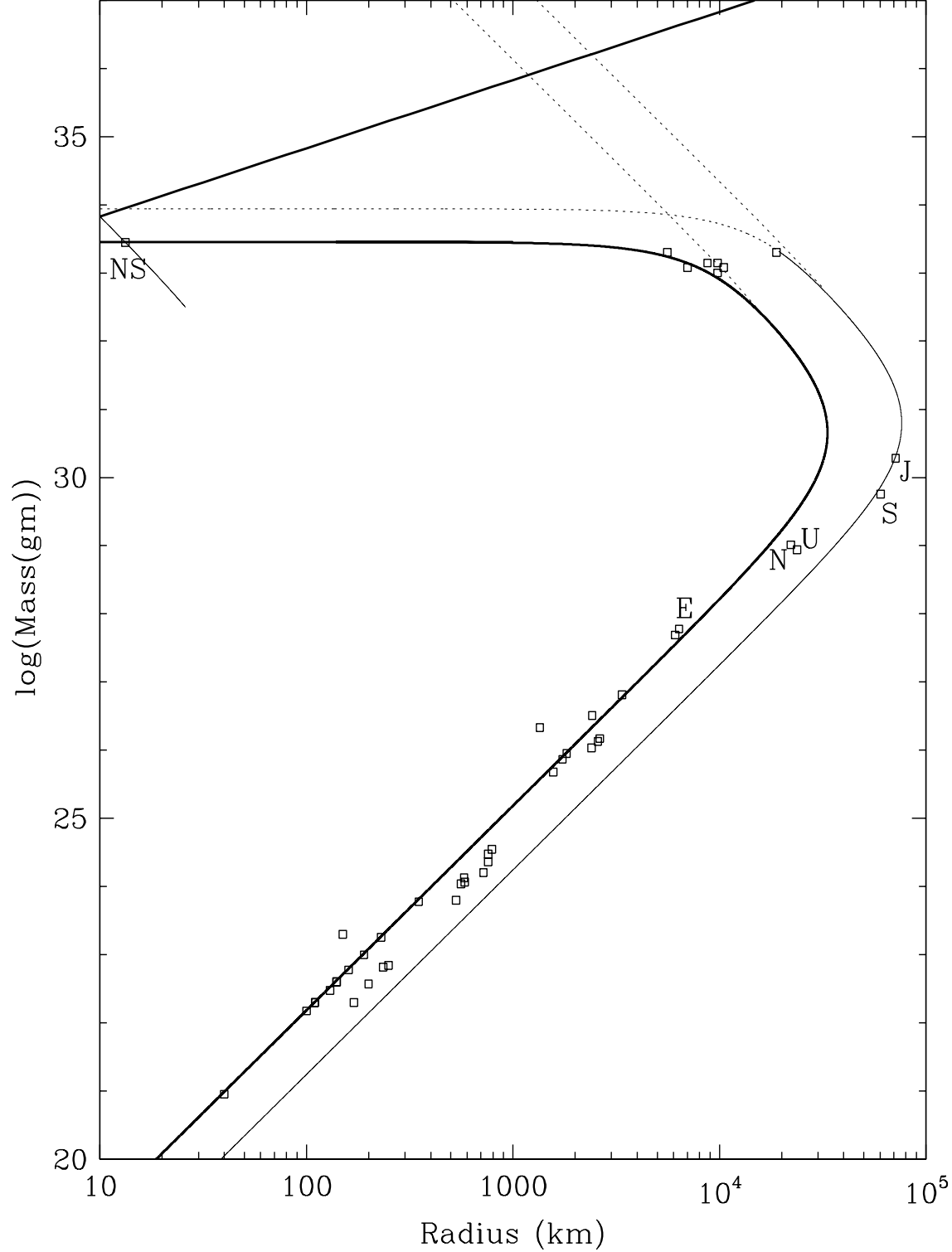
Figure 1: The dotted line gives the non-relativistic approximation that omits the surd and the bracketed expression. The full line gives the full expression for $\rho_o = 3.65 gmcm^{-3}$, $\mu_e = 2$ appropriate for rocky planets and white dwarfs. The lighter line gives the expression for a 25% Helium 75% Hydrogen body or a pristine white dwarf. The squares are solar system bodies supplemented by seven white dwarfs of known radius and one neutron star. The neutron star line is parallel to the dotted lines for non-relativistic white dwarfs but reduced in radius from the pure hydrogen white dwarf by the mass ratio of the neutron and electron.

If we were to ignore all the refinements in III and IV the same formulae hold provided that ρ_o is replaced by $\left(\frac{m_n}{m_e}\right)^3 \rho_o$ whenever we consider neutron stars. Thus the radii are smaller, by the factor $\frac{m_e}{m_n}$, than the white dwarf of the same mass with $\mu_e = 1$. The Chandrasekhar limit is almost unchanged at $5.82 M_\odot$. [1] In practice effects III and IV reduce this to between two and 3.6 solar masses, see Shapiro & Teukolsky. These basic points were understood by Baade & Zwicky (1934) writing soon after the neutron was discovered. Later they took spectra of the Crab Pulsar without realising that it pulsed thirty times a second, but Baade could not interpret the continuous spectrum as, in spite of Schott's (1912) work, Synchotron radiation was not known in astronomy at that time.

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A Ritter's formula and the evaluation of the C_n

$$\frac{1}{\xi^2}\ \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \ \ ,$$

$\theta = 1$, $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$. Let the edge be at $\xi = \xi_1$.

Define the "mass" within ξ by $\mu(\xi) = \int_0^\xi \theta^n \xi^2 d\xi$.

Then $-\xi^2\frac{d\theta}{d\xi} = \mu$ and $\frac{d\mu}{d\xi} = \xi^2\theta^n$,

also $\xi^3\frac{d\theta^{n+1}}{d\xi} = (n+1)\xi^2\theta^n\xi^2\frac{d\theta}{d\xi}/\xi = -(n+1)\frac{\mu}{\xi}\ \frac{d\mu}{d\xi}$.

Thus

$$\int_0^{\xi_1}\xi^3\frac{d\theta^{n+1}}{d\xi}d\xi = -3\int_0^{\xi_1}\theta^{n+1}\xi^2 d\xi = -(n+1)\int_0^{\mu_1}\frac{\mu}{\xi}d\mu \ \ , \qquad (A1)$$

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[1]Using the values of the Fundamental Constants & the Sun's Mass then available Chandrasekhar (1938) got $5.75 M_\odot$

Now write $\psi = \theta + \psi_1$ and take ψ to obey

$$\frac{1}{\xi^2}\;\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = \begin{cases} -\theta^n & \xi \le \xi_1 \\ 0 & \xi > \xi_1 \end{cases} \quad .$$

Furthermore choose ψ_1 so that $\psi \to 0$ at ∞.

$$\frac{d\psi}{d\xi} = \begin{cases} -\frac{\mu}{\xi^2} & \xi \le \xi_1 \\[2mm] -\mu_1/\xi^2 & \xi > \xi_1 \end{cases} \quad .$$

Hence

$$\psi_1 = \mu_1/\xi_1 \qquad \psi = \int_\xi^\infty \frac{\mu}{\xi^2}d\xi \quad , \tag{A2}$$

where

$$\mu = \mu_1 \text{ for } \xi \ge \xi_1 \quad ,$$

$$\int_0^{\xi_1}\xi^2\theta^{n+1}d\xi = \int_0^{\xi_1}\theta\frac{d\mu}{d\xi}d\xi = \int_0^{\xi_1}(\psi - \psi_1)\frac{d\mu}{d\xi}d\xi = \int_0^{\xi_1}\psi\frac{d\mu}{d\xi}d\xi - \frac{\mu_1^2}{\xi_1}$$
using (2)

$$= \int_0^{\xi_1}\int_\xi^\infty \mu\xi^{-2}d\xi\frac{d\mu}{d\xi}d\xi - \frac{\mu_1^2}{\xi} = \int_0^{\mu_1}\frac{\mu}{\xi}d\mu + \int_0^{\mu_1}\frac{\mu}{\xi}d\mu - \frac{\mu_1^2}{\xi_1}$$

$$= 2\int_0^{\mu_1}\frac{\mu}{\xi}d\mu - \frac{\mu_1^2}{\xi_1} = \frac{n+1}{3}\int\frac{\mu d\mu}{\xi} \text{ by (1)}$$

Hence

$$\frac{5-n}{3}\int\frac{\mu d\mu}{\xi} = \frac{\mu_1^2}{\xi_1}$$

which gives Ritter's formula

$$-\mathrm{V} = \frac{3}{5-n}\;\frac{GM^2}{R} \text{ and } \int_0^{\xi_1}\xi^2\theta^{n+1}d\xi = \frac{n+1}{5-n}\;\frac{\mu_1^2}{\xi_1} \quad .$$

Now

$$C_n = \frac{\int_0^1 3\left(\frac{\xi}{\xi_1}\right)^2\theta^{n+1}d\left(\frac{\xi}{\xi_1}\right)}{\left[\int_0^1 3\left(\frac{\xi}{\xi_1}\right)^2\theta^n d\left(\frac{\xi}{\xi_1}\right)\right]^{1+\frac{1}{n}}} = \frac{3^{-\frac{1}{n}}\frac{n+1}{5-n}\frac{\mu_1^2}{\xi_1^4}}{\left(\mu_1/\xi_1^3\right)^{1+\frac{1}{n}}}$$

$$C_n = 3^{-1/n}\;\frac{n+1}{5-n}\mu_1^{1-\frac{1}{n}}\xi_1^{\frac{3}{n}-1}$$

$$\mu_1 = -\left(\xi^2\;\frac{d\theta}{d\xi}\right)_{\xi_1}$$

From tables of polytropes

$$n = 3/2 \quad \xi_1 = 3.65375 \quad \mu_1 = 2.71406$$

$$n = 3 \quad \xi_1 = 6.89685 \quad \mu_1 = 2.01824$$

Hence

$$C_{3/2} = 1.7501$$

$$C_3 = 2.2146$$